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NOTE

Effect of Spatial Resolution on Apparent Sensitivity to Initial Conditions of a Decaying Flow as It Becomes Turbulent

In [1] we obtained numerical solutions for decaying Navier–Stokes flows. Although the higher Reynolds-number solutions indicated sensitive dependence on initial conditions (chaoticity), there may be some question about the effect of spatial resolution on the numerical results. For example, the effect of spatial resolution on sustained Bénard convection was studied in [2], where it was found that spurious chaos can arise in underresolved flows. The complexity of a flow in some cases decreased with improved resolution, although sensitivity to small initial-condition changes appears not to have been determined.

Here, in order to check for spurious chaos and to obtain better solutions for our problem in [1], we explore the effect of improved numerical resolution and numerical method on the chaoticity of solutions for decaying Navier–Stokes flows—that is, for flows described by the Navier–Stokes equations without energy input. The Navier–Stokes equations for an incompressible flow can be written as

$$\frac{\partial u_i}{\partial t} = -\frac{\partial(u_i u_k)}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k\, \partial x_k}, \tag{1}$$

where the pressure is given by the Poisson equation

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_l\, \partial x_l} = -\frac{\partial^2 (u_l u_k)}{\partial x_l\, \partial x_k}. \tag{2}$$

The subscripts can have the values 1, 2, or 3, and a repeated subscript in a term indicates a summation, with the subscript successively taking on the values 1, 2, and 3. The quantity u_i is an instantaneous velocity component, x_i is a space coordinate, t is the time, ρ is the density, ν is the kinematic viscosity, and p is the instantaneous pressure. As in [1] the initial velocity is given by

$$u_i = a_i \cos \mathbf{q}\cdot\mathbf{x} + b_i \cos \mathbf{r}\cdot\mathbf{x} + c_i \cos \mathbf{s}\cdot\mathbf{x}, \tag{3}$$

where

$$\begin{aligned} &a_i = k(2, 1, 1), && b_i = k(1, 2, 1), \\ &c_i = k(1, 1, 2), && q_i = (-1, 1, 1)/x_0, \\ &r_i = (1, -1, 1)/x_0, && s_i = (1, 1, -1)/x_0, \end{aligned} \tag{4}$$

k is a quantity that fixes the initial Reynolds number at $t=0$, and x_0 is one over the magnitude of an initial wavenumber component. The initial pressure is not specified, since it is calculated from Eq. (2). Equations (3) and (4) satisfy continuity, and Eqs. (1) and (2) ensure that continuity is maintained. The boundary conditions are periodic, with a period of $2\pi x_0$.

Cubical computational grids are used for the numerical calculations. For the 32^3 and 64^3 grid-point cases fourth-order spatial differencing and third-order Adams–Moulton predictor–corrector time differencing are used (see [1] for more detail). For 128^3 grid points a pseudospectral method is used for calculating the spatial derivatives [3]. The products of velocity components in the nonlinear terms are obtained before taking Fourier transforms. An isotropic wavenumber truncation of the higher modes is used to eliminate aliasing instabilities [3, Eq. (7.2.19)]. Since a pseudospectral method is of high (approaching infinite) order, the 128^3 grid-point case should give even better resolution than if it had used the fourth-order method. A second-order Adams–Bashforth method is used for the time differencing in the pseudospectral code. As in [1], numerical stability limitations force the timewise resolution (about 20 time steps in the shortest velocity fluctuations) to be good. The emphasis here is therefore on the effect of spatial resolution.

In [4] we showed, by calculating the largest Liapunov exponents, that long-term asymptotic solutions for steadily forced Navier–Stokes turbulence have sensitive dependence on initial conditions. For decaying flows, long-term asymptotic solutions are, of course, stable fixed points in phase space, so that the Liapunov exponents as usually defined, are negative. Thus there is no dependence on initial conditions after a long time. We can, however, consider sensitivity to initial conditions at earlier times. Since it is not clear how Liapunov exponents might be meaningfully defined for decaying flows, we investigate the chaoticity of the latter, as in [1], by simply perturbing the initial conditions a small amount and comparing the perturbed solutions with the unperturbed ones. Here we perturb the a_i, b_i, and c_i in Eq. (4) by 0.1%, while maintaining continuity. For

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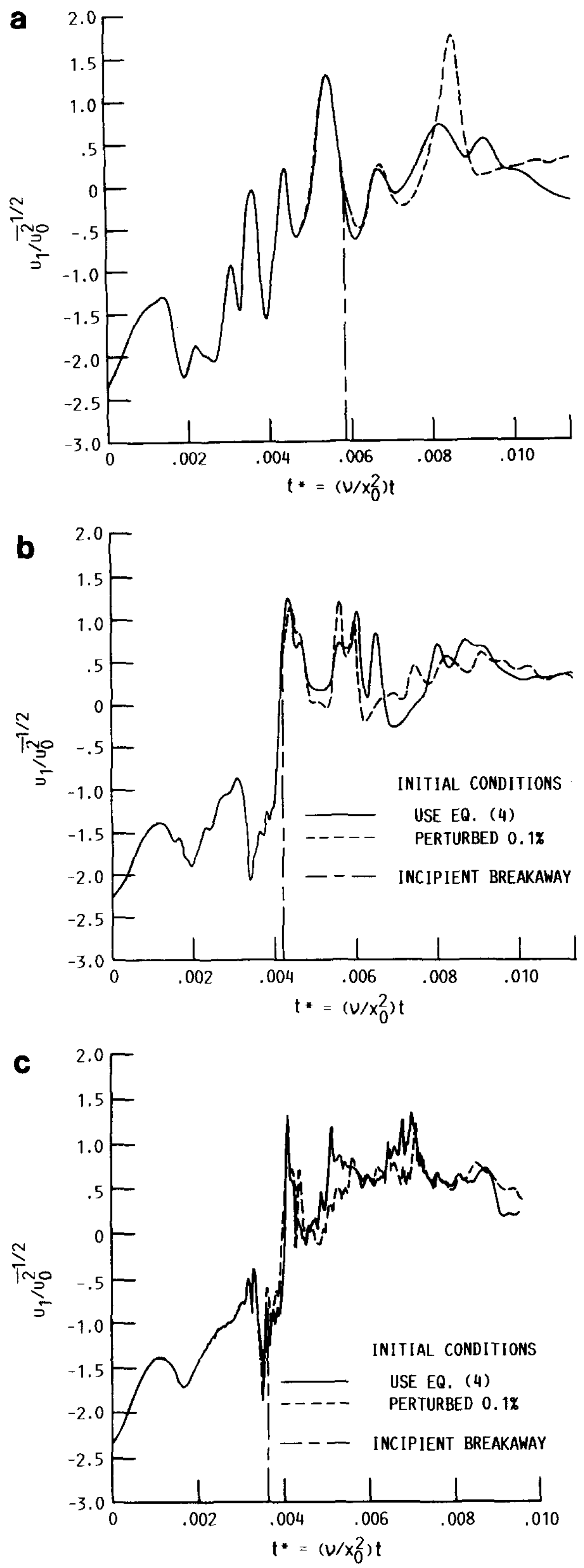

FIG. 1. Calculated evolution of velocity fluctuations at grid center (normalized by initial conditions) for initial Reynolds number $(\overline{u_0^2})^{1/2} x_0/\nu = 1108$ (k in Eq. (4) = 640). Root-mean-square fluctuations are spatially averaged. (a) 32^3 grid points. (b) 64^3 grid points. (c) 128^3 grid points.

example, the a_i are perturbed from $k(2, 1, 1)$ to $k(2.002, 1.001, 1.001)$. The constant k is taken to be 640.

Figure 1 shows the effect of spatial resolution on the indicated sensitivity of the flow to small changes in initial conditions. We take, as a measure of that sensitivity, the value of t^* at which a perturbed solution first shows a definite break with the corresponding unperturbed one. It is clear that improved resolution increases the sensitivity of the solution to small initial-condition changes; the perturbed solution breaks away from the unperturbed one sooner for the more highly resolved cases. Similar results were obtained for other velocity components and at other grid points. These are comforting observations since, if the results were otherwise, the observed chaos in our solutions might be due to inadequate numerical resolution.

Because the time steps in Fig. 1 are generally determined by stability considerations, those steps are smaller for cases where the spatial grid-point spacing is smaller. However, results obtained by using the same time step for all three grid sizes are similar to those shown. In all cases very good time resolution is indicated, since there are at least 20 time steps in the shortest fluctuations.

The results in Fig. 2 indicate that the case with 128^3 grid points is rather well-resolved spatially for $t^* = 0.005$. This is particularly so because of the high-order accuracy of the pseudospectral method used. (The grid points are shown by symbols.) The resolution at earlier times (not shown) is also good because, although the Reynolds number is higher at earlier times, the smaller-scale fluctuations (which are absent in the initial conditions) had not yet developed. The resolution for $t^* = 0.005$ occurs in spite of some steep gradients in the flow. The tendency to form steep gradients at some points in the flow is, of course, a well-known

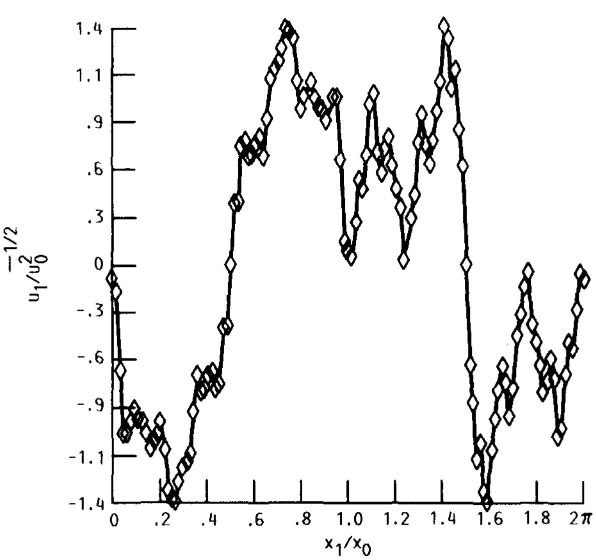

FIG. 2. Calculated spatial variation of velocity fluctuations on a plane through grid center. Symbols are at grid points. Number of grid points, 128^3. $t^* = 0.0050$.

property of turbulent flows and evidently occurs as a steepening effect of the nonlinear terms in the Navier–Stokes equations.

To summarize, grids with 32^3, 64^3, and 128^3 points are used in numerical solutions for a decaying flow. The results indicate that the sensitivity of initially neighboring solutions to small changes in initial conditions increases as the spatial resolution improves. A fourth-order finite-difference method is used for the solutions with 32^3 and 64^3 grid points, and a pseudospectral method is used for 128^3 grid points. The latter solutions appear to be rather well-resolved, in spite of the formation of some steep velocity gradients in the flow.

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